

Ground-state Riemannian metric, cyclic quantum distance, and the quantum criticality in an inhomogeneous Ising spin chain

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(Dated: March 11, 2014)

We investigate the ground-state Riemannian metric and the cyclic quantum distance of an inhomogeneous quantum Ising spin-1/2 chain in a transverse field. This model can be diagonalized by using a general canonical transformation to the fermionic Hamiltonian mapped from the spin system. The ground-state Riemannian metric is derived exactly on a parameter manifold ring S^1 , which is introduced by performing a gauge transformation to the spin Hamiltonian through a twist operator. The ground-state cyclic quantum distance and the second derivative of the ground-state energy are studied in different inhomogeneous exchange coupling parameter region. Particularly, we show that the quantum ferromagnetic phase in the uniform Ising chain can be characterized by an invariant cyclic quantum distance with a constant ground-state Riemannian metric, and this metric will rapidly decay to zero in the paramagnetic phase.

PACS numbers: 64.60.-i, 03.65.Vf, 03.67.-a, 05.70.Fh

I. INTRODUCTION

Quantum phase transitions (QPTs) are driven purely by the quantum fluctuations when a parameter of the Hamiltonian describing the system varies [1–3]. Traditionally, QPTs can be well understood in the framework of the Landau-Ginzburg-Wilson paradigm by resorting to the notions of local order parameter, long range correlations and symmetry breaking. In the past few years, a lot of efforts have been devoted into understanding the QPTs from the information-geometry perspectives [4, 5], such as quantum entanglement [6–9], entanglement entropy [10–15], quantum discord [16–20], quantum fidelity and fidelity susceptibility [21–31], Berry phase [32–40] and the quantum geometric tensor [41–49].

Generally, QPTs can be witnessed by some qualitative changes of the ground-state properties when some parameters of the Hamiltonian across the quantum critical point (QCP). The underlying physical mechanism lies in the fact that the different phases are unconnected by the adiabatic evolution of the ground state. In the vicinity of the QCP, the ground state driven by the parameters of the Hamiltonian will lead to an avoided energy-level crossing between the ground state and the first excited state, where the adiabatic evolution can be destroyed by a vanishing energy gap as the system size tends to infinity. From perspective of the differential geometry of the ground state, a monopole as a gapless point in the Hamiltonian parameters space will generate some interesting effect on the ground-state local or topological properties, and these properties can be captured by some local quantities, i.e., the fidelity susceptibility and the Berry curvature; or by some topological quantum numbers, i.e., the Chern number [50, 51], Z_2 number [52–55], and recently,

the Euler number of the Bloch states manifold has been proposed [56, 57].

Recently, the concept of ground-state quantum geometric tensor has been introduced to analyze the QPTs. What is surprising is that the two approaches of the ground-state Berry curvature and the fidelity susceptibility as a witness to QPTs are unified. Specifically, the real part of the QGT is a Riemannian metric defined over the parameter manifold, while the imaginary part is the Berry curvature which flux give rise to the Berry phase. The Riemannian metric is recognized as the essential part of the fidelity susceptibility. Generally, the Riemannian metric and the Berry curvature will exhibit some singularity or scaling behavior in the quantum critical region under the thermodynamic limit. Particularly, a scaling analysis of the ground-state quantum geometric tensor in the vicinity of the critical points has been performed. So far, these approaches have been applied to detect the phases boundaries in various systems.

In this work, we propose a cyclic quantum distance of the ground state to detect the QPTs in a transverse field inhomogeneous Ising spin-1/2 chain, in which the nearest-neighbor exchange interactions will take alternating parameters between the neighbor sites. This model can be solved exactly by introducing a general canonical transformation to diagonalize the fermionic Hamiltonian mapped from the spin Hamiltonian by the Jordan-Wigner transformation. In our scheme, an extra local gauge transformation is performed to the spin system by a twist operator, which endows the Hamiltonian of the system with a topology of a ring S^1 without changing its energy spectrum. We obtain the exact expression of the ground-state Riemannian metric and study the cyclic quantum distance of the ground state on the parameter

manifold. We study extensively the ground-state Riemannian metric in different parameter region of the inhomogeneous Ising chain. Particularly, we show that the quantum ferromagnetic phase in the uniform Ising chain can be marked by an invariant cyclic quantum distance of the ground state, and the distance decay to zero rapidly in the paramagnetic phase.

II. THE MODEL

Let us consider an inhomogeneous Ising spin-1/2 chain, which consists of N cells with two sites in each cell, and in an external magnetic field. The Hamiltonian reads

$$\mathcal{H} = - \sum_{l=1}^N [J_a \sigma_{l,a}^x \sigma_{l,b}^x + J_b \sigma_{l,b}^x \sigma_{l+1,a}^x + h(\sigma_{l,a}^z + \sigma_{l,b}^z)] \quad (1)$$

where $\sigma_{l,m}^\alpha$ ($\alpha = x, y, z$; $m = a, b$) are the local Pauli operators, J_a (J_b) is the exchange coupling, h is the external field and the periodic boundary condition (PBC) has been assumed. A similar inhomogeneous XY spin model has been investigated in Ref.[37], and here we give a brief discussion for the completeness of this work. First, we subject the system to a local gauge transformation $\mathcal{H}(\varphi) = \mathcal{D}_z(\varphi) \mathcal{H} \mathcal{D}_z^\dagger(\varphi)$ by a twist operator $\mathcal{D}_z(\varphi) = \prod_{l=1}^N \exp[i\varphi(\sigma_{l,a}^z + \sigma_{l,b}^z)/2]$, which in fact makes the system rotate on the spin along the z -direction. It can be verified that $\mathcal{H}(\varphi)$ is π periodic in the parameter φ . Considering the unitarity of the twist operator $\mathcal{D}_z(\varphi)$, the energy spectrum and critical behavior of the system are obviously independent with the parameter φ . The spin Hamiltonian can be mapped exactly on a spinless fermion model through the Jordan-Wigner transformation $S_{l,a}^+ = C_{l,a}^\dagger \exp i\pi \sum_{l'=1}^{l-1} (C_{l',a}^\dagger C_{l',a} + C_{l',b}^\dagger C_{l',b})$ and $S_{l,b}^+ = C_{l,b}^\dagger \exp i\pi \sum_{l'=1}^{l-1} (C_{l',a}^\dagger C_{l',a} + C_{l',b}^\dagger C_{l',b}) + i\pi C_{l,a}^\dagger C_{l,a}$, where $S_{l,a(b)}^\pm = (\sigma_{l,a(b)}^x \pm i\sigma_{l,a(b)}^y)/2$ are the spin ladder operators, and $C_{l,a(b)}$ are the fermion operators. The Hamiltonian $\mathcal{H}(\varphi)$ is transformed into

$$\begin{aligned} H(\varphi) = & - \sum_{l=1}^N J_a \left(C_{l,a}^\dagger C_{l,b} + e^{-i2\varphi} C_{l,a}^\dagger C_{l,b}^\dagger + H.c. \right) \\ & + J_b \left(C_{l,b}^\dagger C_{l+1,a} + e^{-i2\varphi} C_{l,b}^\dagger C_{l+1,a}^\dagger + H.c. \right) \\ & + 2h(C_{l,a}^\dagger C_{l,a} + C_{l,b}^\dagger C_{l,b} - 1). \end{aligned} \quad (2)$$

Note that the PBC on the spin degrees of freedom $\sigma_{N+1,m}^\alpha = \sigma_{1,m}^\alpha$ ($\alpha = x, y, z$; $m = a, b$) imply that $C_{N+1,m} = e^{i\pi N_F} C_{1,m}$, where $N_F := \sum_{l'=1}^N \sum_{m'=a,b} C_{l',m'}^\dagger C_{l',m'}$ denotes the total fermion number. Thus the boundary conditions on the fermionic system will obey PBC or anti-PBC depending on whether N_F is even or odd. However, the differences between the two boundary conditions are negligible in the thermodynamic limit. Without loss of generality, we take the PBC

on the fermionic system, which means $C_{N+1,m} = C_{1,m}$. Second, applying the following Fourier transformation $C_{l,a} = \frac{1}{\sqrt{N}} \sum_k e^{ikR_{la}} a_k$ and $C_{l,b} = \frac{1}{\sqrt{N}} \sum_k e^{ik(R_{la}+a)} b_k$ to the Hamiltonian $H(\varphi)$, where $k = (2\pi/2Na)n$, ($n = -\frac{N-1}{2}, -\frac{N-1}{2}+1, \dots, \frac{N-1}{2}$) and R_{la} ($R_{lb} = R_{la} + a$) is the coordinate of site a (b) on the l -th cell in the lattice with the lattice parameter $2a$. Now, the Hamiltonian $\mathcal{H}(\varphi)$ in the momentum space reads

$$\begin{aligned} H(\varphi) = & - \sum_k 2h(a_k^\dagger a_k + b_k^\dagger b_k - 1) \\ & + [(J_a e^{ika} + J_b e^{-ika}) a_k^\dagger b_k + H.c.] \\ & - [(J_a e^{i2\varphi+ika} - J_b e^{i2\varphi-ika}) a_{-k} b_k + H.c.] \end{aligned} \quad (3)$$

This Hamiltonian $H(\varphi)$ can be exactly diagonalized as

$$H(\varphi) = \sum_{q=\gamma,\eta,\mu,\nu} \sum_k \Lambda_{q,k} \left(q_k^\dagger q_k - \frac{1}{2} \right), \quad (4)$$

by using the following canonical transformation, i.e.,

$$\begin{aligned} \gamma_k &= \frac{1}{\sqrt{2}} (e^{i2\varphi} \cos \frac{\theta_k}{2} a_k + e^{i\delta_k} e^{-i\sigma_k} \sin \frac{\theta_k}{2} a_{-k}^\dagger \\ &\quad - e^{i2\varphi} e^{i\delta_k} \cos \frac{\theta_k}{2} b_k + e^{i\sigma_k} \sin \frac{\theta_k}{2} b_{-k}^\dagger), \\ \eta_k &= \frac{1}{\sqrt{2}} (-e^{-i\delta_k} e^{i\sigma_k} \sin \frac{\theta_k}{2} a_k + e^{-i2\varphi} \cos \frac{\theta_k}{2} a_{-k}^\dagger \\ &\quad + e^{i\sigma_k} \sin \frac{\theta_k}{2} b_k + e^{-i2\varphi} e^{-i\delta_k} e^{2i\sigma_k} \cos \frac{\theta_k}{2} b_{-k}^\dagger), \\ \mu_k &= \frac{1}{\sqrt{2}} (e^{i2\varphi} \cos \frac{\beta_k}{2} a_k - e^{i\delta_k} e^{-i\sigma_k} \sin \frac{\beta_k}{2} a_{-k}^\dagger \\ &\quad + e^{i2\varphi} e^{i\delta_k} \cos \frac{\beta_k}{2} b_k + e^{i\sigma_k} \sin \frac{\beta_k}{2} b_{-k}^\dagger), \\ \nu_k &= \frac{1}{\sqrt{2}} (e^{-i\delta_k} e^{i\sigma_k} \sin \frac{\beta_k}{2} a_k + e^{-i2\varphi} \cos \frac{\beta_k}{2} a_{-k}^\dagger \\ &\quad + e^{i\sigma_k} \sin \frac{\beta_k}{2} b_k - e^{-i2\varphi} e^{-i\delta_k} e^{2i\sigma_k} \cos \frac{\beta_k}{2} b_{-k}^\dagger), \end{aligned} \quad (5)$$

where

$$\begin{aligned} \cos \theta_k &= \frac{2h - M_k}{\sqrt{(2h - M_k)^2 + N_k^2}}, \\ \cos \beta_k &= \frac{2h + M_k}{\sqrt{(2h + M_k)^2 + N_k^2}}, \\ M_k &= \sqrt{J_a^2 + J_b^2 + 2J_a J_b \cos 2ka}, \\ N_k &= \sqrt{J_a^2 + J_b^2 - 2J_a J_b \cos 2ka}, \\ \delta_k &= \arg(J_a e^{ika} + J_b e^{-ika}), \\ \sigma_k &= \arg(J_a e^{ika} - J_b e^{-ika}), \end{aligned} \quad (6)$$

and the quasiparticle energy spectrums are

$$\begin{aligned}\Lambda_{\gamma k} &= -\frac{1}{2}(2h - M_k) - \frac{1}{2}\sqrt{(2h - M_k)^2 + N_k^2}, \\ \Lambda_{\eta k} &= -\frac{1}{2}(2h - M_k) + \frac{1}{2}\sqrt{(2h - M_k)^2 + N_k^2}, \\ \Lambda_{\mu k} &= -\frac{1}{2}(2h + M_k) - \frac{1}{2}\sqrt{(2h + M_k)^2 + N_k^2}, \\ \Lambda_{\nu k} &= -\frac{1}{2}(2h + M_k) + \frac{1}{2}\sqrt{(2h + M_k)^2 + N_k^2}.\end{aligned}\quad (7)$$

III. CYCLIC QUANTUM DISTANCE AND GROUND-STATE RIEMANNIAN METRIC

Now, we focus on the geometric properties of the ground state. The Hamiltonian $H(\varphi)$ in Eq. (4) has been diagonalized in the set of quasiparticle number operators, which allows us to determine all the eigenvalues and eigenvectors. We note that the energy spectrums $\Lambda_{\eta k} \geq 0$, $\Lambda_{\nu k} \geq 0$ and $\Lambda_{\gamma k} \leq 0$, $\Lambda_{\mu k} \leq 0$. The ground state, denoted as $|GS(\varphi)\rangle$, corresponds to the state with the lowest energy, which consists of state with no η and ν fermions occupied but with γ and μ fermions occupied. Explicitly, the ground state can be constructed as follows

$$|GS(\varphi)\rangle = \mathcal{N}^{-\frac{1}{2}} \prod_{k>0} \left(\gamma_{-k}^\dagger \gamma_k^\dagger \mu_{-k}^\dagger \mu_k^\dagger \eta_{-k} \eta_k \nu_{-k} \nu_k \right) |0\rangle, \quad (8)$$

where $\mathcal{N}^{-\frac{1}{2}}$ is the normalized factor, and $|0\rangle$ are the vacuum states of fermionic operators a_k and b_k , respectively. It is easy to check that $\eta_k|GS(\varphi)\rangle = 0$, $\nu_k|GS(\varphi)\rangle = 0$ and $\gamma_k^\dagger|GS(\varphi)\rangle = 0$, $\mu_k^\dagger|GS(\varphi)\rangle = 0$ for all k . The corresponding ground-state energy E_g is

$$\begin{aligned}E_g &= \sum_k \left(\frac{1}{2}\Lambda_{\gamma k} + \frac{1}{2}\Lambda_{\mu k} - \frac{1}{2}\Lambda_{\eta k} - \frac{1}{2}\Lambda_{\nu k} \right) \\ &= \sum_k -\frac{1}{2}\sqrt{(2h - M_k)^2 + N_k^2} \\ &\quad -\frac{1}{2}\sqrt{(2h + M_k)^2 + N_k^2}.\end{aligned}\quad (9)$$

Now, we introduce the notion of quantum geometric tensor of the ground state $|GS(\varphi)\rangle$ on the Hamiltonian parameter φ manifold. It can be verified that the quantum geometric tensor can be derived from a gauge invariant distance between two ground states on the $U(1)$ line bundle induced by the quantum adiabatic evolution of ground states $|g(\varphi)\rangle$ in parameter φ space. The quantum distance dS between two ground states $|GS(\varphi)\rangle$ and $|GS(\varphi + \delta\varphi)\rangle$ is given by $dS^2 = \langle \partial_\varphi GS(\varphi) | \partial_\varphi GS(\varphi) \rangle d\varphi^2$. Note that the term $|\partial_\varphi GS(\varphi)\rangle$ can be decomposed as $|\partial_\varphi GS(\varphi)\rangle = |D_\varphi GS(\varphi)\rangle + [1 - \mathcal{P}(\varphi)] |\partial_\varphi GS(\varphi)\rangle$, where $\mathcal{P}(\varphi) = |GS(\varphi)\rangle \langle GS(\varphi)|$ is the projection operator and $|D_\varphi GS(\varphi)\rangle = \mathcal{P}(\varphi) |\partial_\varphi GS(\varphi)\rangle$ is the covariant derivative of $|GS(\varphi)\rangle$ on the $U(1)$ line bundle. Under the condition of the quantum adiabatic evolution, the evolution from

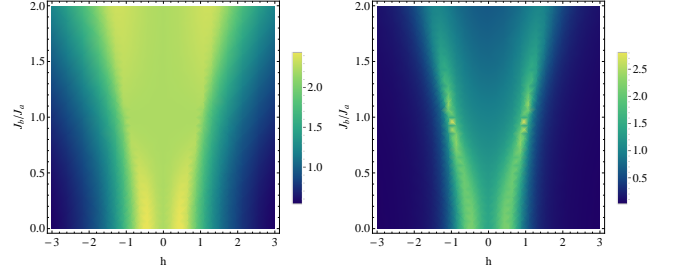


FIG. 1: (color online) (Left) The cyclical quantum distance $l(0, \pi)$ as a function of J_b/J_a and h with the fixed parameters $J_a = 1$, and the system sizes $N \rightarrow \infty$; (Right) The second derivative of the ground-state energy E_g/N with respect to h , as a function of J_b/J_a and h with the fixed parameters $J_a = 1$, and the system sizes $N = 1001$.

$|GS(\varphi)\rangle$ to $|GS(\varphi + \delta\varphi)\rangle$ will undergo a parallel transport in the sense of Levi-Civita from φ to $\varphi + \delta\varphi$ on the parameter manifold, and hence we have $|D_\varphi GS(\varphi)\rangle = 0$. Finally, we can obtain the quantum distance as $dS^2 = \langle \partial_\varphi GS(\varphi) | (1 - |GS(\varphi)\rangle \langle GS(\varphi)|) | \partial_\varphi GS(\varphi) \rangle d\varphi^2$. The quantum geometric tensor is given by

$$Q_{\varphi\varphi} = \langle \partial_\varphi GS(\varphi) | (1 - |GS(\varphi)\rangle \langle GS(\varphi)|) | \partial_\varphi GS(\varphi) \rangle. \quad (10)$$

Substituting Eq. (8) into Eq. (10), we can derive the concrete expression of $Q_{\varphi\varphi}$. Obviously, the straightforward calculation is tedious. We note that the result can be derived concisely from the following consideration. To begin with, the term $\langle g(\varphi) | \partial_\varphi g(\varphi) \rangle$ of $Q_{\varphi\varphi}$ can be write as

$$\begin{aligned}\langle g(\varphi) | \partial_\varphi g(\varphi) \rangle &= \mathcal{N}^{-1} \prod_{k,j>0} \langle 0 | (\nu_j^\dagger \nu_{-j}^\dagger \eta_j^\dagger \eta_{-j}^\dagger \mu_j \mu_{-j} \gamma_j \gamma_{-j}) \partial_\varphi \\ &\quad \left(\gamma_{-k}^\dagger \gamma_k^\dagger \mu_{-k}^\dagger \mu_k^\dagger \eta_{-k} \eta_k \nu_{-k} \nu_k \right) |0\rangle.\end{aligned}\quad (11)$$

We note that each term of $\langle 0 | \gamma_k \partial_\varphi \gamma_k^\dagger |0\rangle$, $\langle 0 | \gamma_{-k} \partial_\varphi \gamma_{-k}^\dagger |0\rangle$ in Eq. (11) yield the same results as $-2i \cos^2 \frac{\theta_k}{2}$, and $\langle 0 | \mu_k \partial_\varphi \mu_k^\dagger |0\rangle$, $\langle 0 | \mu_{-k} \partial_\varphi \mu_{-k}^\dagger |0\rangle$ will yield the results as $-2i \cos^2 \frac{\beta_k}{2}$, meanwhile the other terms yield the results as 0. Finally, we can get

$$\langle GS(\varphi) | \partial_\varphi GS(\varphi) \rangle = \sum_{k>0} -2i \left(\cos^2 \frac{\theta_k}{2} + \cos^2 \frac{\beta_k}{2} \right). \quad (12)$$

Here, we can define a connection as $\mathcal{A}_\varphi = i \langle GS(\varphi) | \partial_\varphi GS(\varphi) \rangle$, which is exactly the Berry-Simon connection of the ground state on the parameter φ manifold. In order to calculate the term of $\langle \partial_\varphi GS(\varphi) | \partial_\varphi GS(\varphi) \rangle$, we note that $\langle \partial_\varphi GS(\varphi) | \partial_\varphi GS(\varphi) \rangle = -\langle GS(\varphi) | \partial_\varphi \partial_\varphi GS(\varphi) \rangle$ because $\partial_\varphi \langle GS(\varphi) | \partial_\varphi GS(\varphi) \rangle = 0$ (see Eq. (12)), and so we have

$$\begin{aligned}
\langle \partial_\varphi g(\varphi) | \partial_\varphi g(\varphi) \rangle &= -\mathcal{N}^{-1} \prod_{k,j>0} \langle 0 | (\nu_j^\dagger \nu_{-j}^\dagger \eta_j^\dagger \eta_{-j}^\dagger \mu_j \mu_{-j} \gamma_j \gamma_{-j}) \partial_\varphi \partial_\varphi \left(\gamma_{-k}^\dagger \gamma_k^\dagger \mu_{-k}^\dagger \mu_k^\dagger \eta_{-k} \eta_k \nu_{-k} \nu_k \right) | 0 \rangle \\
&= \sum_{k>0} \sum_{j>0} \left(2i \cos^2 \frac{\theta_k}{2} + 2i \cos^2 \frac{\beta_k}{2} \right) \left(-2i \cos^2 \frac{\theta_j}{2} - 2i \cos^2 \frac{\beta_j}{2} \right) \\
&\quad - \sum_{k>0} 4 \left(\cos^4 \frac{\theta_k}{2} + \cos^4 \frac{\beta_k}{2} \right) + \sum_{k>0} 4 \left(\cos^2 \frac{\theta_k}{2} + \cos^2 \frac{\beta_k}{2} \right). \tag{13}
\end{aligned}$$

Substituting Eq. (12) and Eq. (13) into Eq. (10), we can obtain the QGT as

$$\begin{aligned}
Q_{\varphi\varphi} &= \langle \partial_\varphi g(\varphi) | \partial_\varphi g(\varphi) \rangle - \langle \partial_\varphi g(\varphi) | g(\varphi) \rangle \langle g(\varphi) | \partial_\varphi g(\varphi) \rangle \\
&= \sum_{k>0} \sin^2 \theta_k + \sin^2 \beta_k. \tag{14}
\end{aligned}$$

Now let us focus on the characterization of the geometric properties of the ground state. In our approach, the ground state $|GS(\varphi)\rangle$ is defined in a $U(1)$ line bundle located over a one dimensional parameter manifold S^1 , and hence the Riemannian metric as the real part of the QGT is just $Q_{\varphi\varphi}$ itself. As we discussed above, the ground-state Riemannian metric provide us a gauge invariant distance measurement of the ground state on the parameter φ manifold. The quantum distance l between two ground states $|GS(\varphi_A)\rangle$ and $|GS(\varphi_B)\rangle$ is given by

$$l(\varphi_A, \varphi_B) = \int_{\varphi_A}^{\varphi_B} \sqrt{\sum_{k>0} \sin^2 \theta_k + \sin^2 \beta_k} d\varphi. \tag{15}$$

To have an explicit view of the dependence of the Riemannian metric on the system size, we can perform a scaling transforming to the $Q_{\varphi\varphi}$ and denote the Riemannian metric as $g = Q_{\varphi\varphi}/L^d = \frac{1}{N} \sum_{k>0} (\sin^2 \theta_k + \sin^2 \beta_k)$, where $L^d = N$ as the number of the sites and here $d = 1$ is the dimension of the system. To study the quantum criticality, we are interested in the properties under the thermodynamic limit when the system size $N \rightarrow \infty$, and we have the Riemannian metric

$$\begin{aligned}
g &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k>0} (\sin^2 \theta_k + \sin^2 \beta_k) \\
&= \frac{1}{2\pi} \int_0^\pi \frac{(2h^2 + J_a^2 + J_b^2)(J_a^2 + J_b^2 - 2J_a J_b \cos k)}{4h^4 + (J_a^2 + J_b^2)^2 - 8h^2 J_a J_b \cos k} dk,
\end{aligned}$$

where the summation $\frac{1}{N} \sum_{k>0}$ has been replaced by the integral $\frac{1}{2\pi} \int_0^\pi dk$. Obviously, the quantum distance $l(0, \pi)$ for a cyclical evolution from $|GS(0)\rangle$ to $|GS(\pi)\rangle$ is given by $l(0, \pi) = \int_0^\pi d\varphi \int_0^\pi \frac{dk}{2\pi} (\sin^2 \theta_k + \sin^2 \beta_k)$. In Fig. 1(Left), we plot the cyclic quantum distance $l(0, \pi)$ as a function of h and $\alpha = J_b/J_a$ with the system size $N \rightarrow \infty$. As a comparison, we also provide a numeric results of the second derivative of the ground-state energy

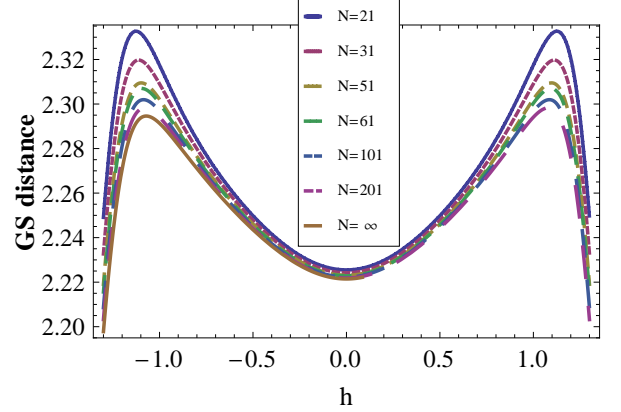


FIG. 2: (color online) The cyclical quantum distance $l(0, \pi)$ as a function h with the fixed parameters $J_a = 1, J_b = 1.5$, and with different system sizes.

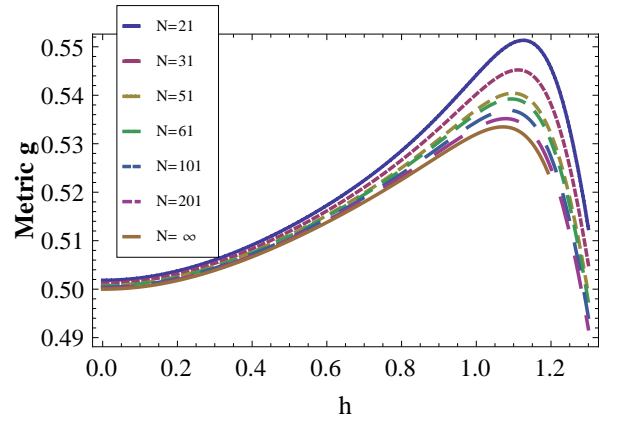


FIG. 3: (color online) The Riemannian metric g as a function of h with the fixed parameters $J_a = 1, J_b = 1.5$, and with different system sizes.

with respect to h , as a function of J_b/J_a and h with the system sizes $N = 1001$ (see Fig. 1(Right)). The cyclic quantum distance $l(0, \pi)$ and the Riemannian metric g as a function of the external field h with the fixed parameters $J_a = 1, J_b = 1.5$ are shown in Fig. 2 and Fig. 3, respectively. In the region of inhomogeneous spin exchange coupling $J_a \neq J_b$, the cyclic quantum distance

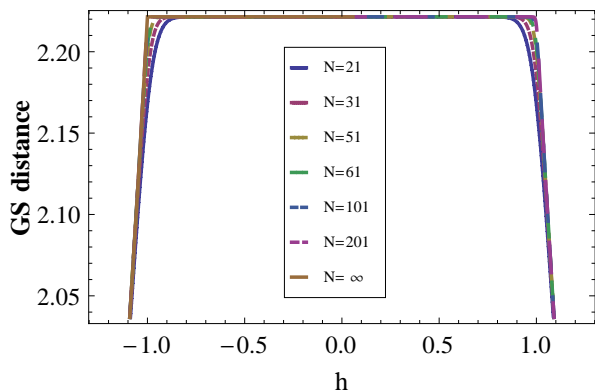


FIG. 4: (color online) The cyclical quantum distance $l(0, \pi)$ as a function h with the fixed parameters $J_a = J_b = 1$, and with different system sizes.

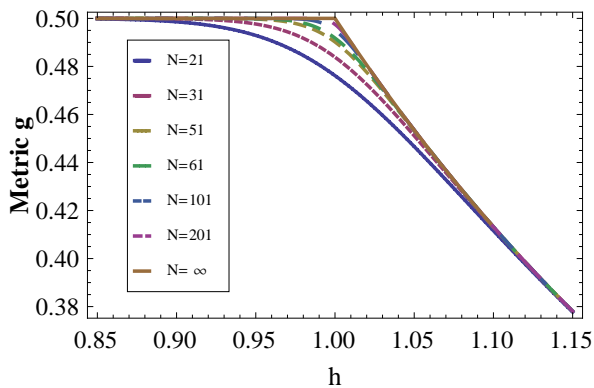


FIG. 5: (color online) The Riemannian metric g as a function of h with the fixed parameters $J_a = J_b = 1$, and with different system sizes.

and the Riemannian metric has a similar trend with the second derivative of the ground-state energy. It is worth reminding that, in the uniform exchange coupling case $J_a = J_b = 1$ and system size $N \rightarrow \infty$, the ground-state Riemannian metric g (see Eq. (16)) can be exactly solved as

$$g = \frac{1}{2\pi} \int_0^\pi \frac{(h^2 + 1)(1 - \cos k)}{h^4 - 2h^2 \cos k + 1} dk, \\ = \begin{cases} \frac{1}{2} & \text{if } |h| \leq 1 \\ \frac{1}{2h^2} & \text{otherwise} \end{cases}, \quad (17)$$

which leads to an invariant ground-state cyclical distance $l = \pi/\sqrt{2}$ in the ferromagnetic phase, and $l = \pi/(\sqrt{2}h)$

in the paramagnetic phase. In Fig. 4 and Fig. 5, we show the properties of the cyclic quantum distance and the Riemannian metric in the vicinity of the critical points with fixed parameters $J_a = J_b = 1$ and different system size N . As shown in Fig. 4 and Fig. 5, the cyclic quantum distance and the Riemannian metric of the ground state in the ferromagnetic phase is in close to the constants of 2.22144 and 0.5, respectively, with the increase of the system size N . In the thermodynamic limit $N \rightarrow \infty$, the first derivative of the cyclic distance and the metric are discontinuous in the critical point.

IV. CONCLUSION

In summary, we study the geometric properties of the ground state of a two-period inhomogeneous quantum Ising chain in a transverse field. Particularly, we introduce an extra local gauge transformation to the spin system by a twist operator, which endows the Hamiltonian of the system with a topology of a ring S^1 without changing its energy spectrum. On the parameter manifold, we derive the exact expression of the ground-state Riemannian metric and define a cyclic quantum distance of the ground state. We study extensively the ground-state Riemannian metric and the cyclic quantum distance in different parameter region of the two-period inhomogeneous Ising chain. Furthermore, we show that the ferromagnetic phase of a uniform Ising chain can be characterized by a invariant cyclic ground-state quantum distance, and in the paramagnetic phase the distance will decay rapidly to zero. This approach provides a interesting description on the geometric properties of the ground state in addition to the ground-state Berry phase approach. We hope that the current work will raise renewed interest in the understanding of the geometric nature of the ground state in quantum condensed-matter systems.

V. ACKNOWLEDGMENTS

This work was supported by the special foundation for theoretical physics research Program of China under grants No. 11347131, the NKBRSCF under Grants No. 2009CB930701, No. 2010CB922904, No. 2011CB921502, and No. 2012CB821300, NSFC under Grants Nos. 10934010, 11001263, 11004007, and NSFC-RGC under Grants No. 11061160490 and No. 1386-N-HKU748/10, NSF of Beijing under Grant No. 1132016 and Beijing Nova program No. xx2013029.

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